A VARIABLE SYSTEM OF SEVENS ON TWO TWISTED CUBIC CURVES

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Seven points chosen at random on a twisted cubic curve, like six points on a conic in the plane, give rise to a distinctive theorem; for as five points determine a conic, so the twisted cubic is determined by six points. In the case of the conic, this is the theorem of the Pascal hexagon, six points in a definite order leading to a definite line. Conic and line remaining fixed, the hexagon may vary with four degrees of freedom. In the case of the twisted cubic, not a mere sequence of the seven points, but an arrangement of them in seven triads, determines seven planes, and the theorem states that these planes are all osculated by a second twisted cubic curve. So much was established by a direct proof in these PROCEEDINGS in August, 1915; but the question of variability, whether the points and planes are free to move while the two curves remain fixed, was not examined. Now it is found that *the system is variable with one degree of freedom.* Full proof is contained in a paper soon to appear in the *Transactions of the American Mathematical Society*. The following is an outline.

Every twisted cubic C_3 is a rational curve, and the homogeneous coordinates of its points are cubic functions of a variable parameter:

$$x_1 = f_1(\lambda),\, x_2 = f_2(\lambda),\quad \ldots\quad, x_4 = f_4(\lambda).$$

In the same way the osculating planes of any second cubic curve K_3 are represented by cubic functions of a second parameter:

$$u_1 = g_1(\mu),\, u_2 = g_2(\mu),\quad \ldots\quad, u_4 = g_4(\mu).$$

Every point of the first may be put in relation to the three planes of the other that pass through it by the equation

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0,$$

or

$$\sum_1^4 f_i(\lambda) \cdot g_i(\mu) \equiv 0,$$

of the third degree in λ and also in μ. Conversely, every bicubical relation, or (3,3) correspondence, may be interpreted as such a point-to-plane relation between points of an arbitrary cubic C_3 and osculat-

ing planes of a second. This second curve K_3, however, is completely determined by the first, C_3, and the relation

$$F(\lambda, \mu) = \Sigma f_i(\lambda) \,.\, g_i(\mu) = 0;$$

for there is only one way of expressing a given bicubical function $F(\lambda, \mu)$ linearly in terms of four independent cubics $f_1(\lambda), f_2(\lambda), f_3(\lambda), f_4(\lambda)$.

Such a bicubical function $F(\lambda, \mu)$ contains 15 arbitrary coefficients, besides one multiplicative constant. If it relates cubic curves whose relative situation is like that of the C_3 and K_3 mentioned above, we may say that the relation $F(\lambda, \mu)$ admits a solution of period 7, or briefly, that it admits a Δ_7. A special kind of (3, 3) relation is that which factors into three (1, 1) relations or projectivities:

$$F(\lambda, \mu) \equiv \varphi(\lambda, \mu) \,.\, \psi(\lambda, \mu) \,.\, \chi(\lambda, \mu),$$

the triply bilinear relation. Of this special kind there is a sub-species which admits a Δ_7.

To prove the theorem stated above, viz., *that every* (3, 3) *relation which admits one* Δ_7 *must necessarily admit a simple infinity of* Δ_7*'s*, I proceed by counting the number of free constants in each of the four classes of (3.3) relations which have just now been noticed.

The first class, the general (3, 3) relation, contains 15 constants, including 3 that might have been deducted for linear transformation of either λ or μ. The second class, that admitting one Δ_7 (or more than one), contains the three constants of a linear transformation and apparently 7 others, since according to the former theorem cited above the 7 points on the C_3 can be chosen at random. If however the presence of one Δ_7 should imply ∞^1 others, the number of free constants would reduce to 9:- call the number 10 − R, where $R = 0$ or 1. The third class contains 9 constants, 3 for each of the collineations involved. The use of the fourth class is probably novel, at least in this connection; it contains 3 constants. The proof of this is the essential part of the demonstration.

The argument is now most easily stated geometrically. In a linear (flat) space of 15 dimensions, two contained algebraic varieties or spreads of s and of k dimensions respectively must have in common a spread of *at least* $s + k - 15$ dimensions. Here $s = 10 - R$, $k = 9$, and the common part or intersection is of 3 dimensions. Hence $10 - R + 9 - 15 \overset{=}{<} 3$, or $1 \overset{=}{<} R$. But we had $R \overset{=}{<} 1$, therefore $R = 1$, as asserted in the theorem.